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A statistical model of the longitudinal dispersion process in turbulent flow in a channel

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SUMMARY

The important practical problem of the dispersion of a passive contaminant in a fluid flowing through a pipe or channel of uniform cross-section is usually analysed in terms of the distribution of concentration. In this paper however a different though approximate approach is adopted which both illustrates the essential statistical nature of the process and may be quicker to employ when approximate answers are acceptable in a practical problem. A simple statistical model is proposed for the motion of a single molecule of contaminant and leads to an expression for the covariance of the velocity of the molecule in terms of the fluid velocity, and hence to a value of Taylor's longitudinal diffusivity. The model is applied to two simple flows in a channel, one of which illustrates the effect of the viscous sub-layer. Despite the number of simplifying assumptions made in constructing the model it gives results which are close to those obtained by conventional means. Ways in which the model could be adapted to give even better results are discussed.

1. Introduction

Consider fluid in turbulent flow through a straight pipe or channel of uniform cross-section and suppose that at some time, say $t=0$, some of the fluid is marked in such a way that it is dynamically indistinguishable from the unmarked fluid. It is of great interest and importance to know how the cloud of marked fluid disperses subsequently. Assuming that the molecules of marked fluid move independently the dispersion can be described in terms of the statistical properties of the motion of a *single* molecule or in terms of the distribution of concentration C within the cloud — which contains *many* molecules. Although the two approaches are consistent most work has used the latter approach because of the simplicity and accuracy of the equation governing C when Fick's Law is assumed to hold. In this paper on the other hand a simple statistical model for the motion of a single molecule of marked fluid is proposed and it is shown that in the cases considered it gives results very similar to those obtained from an analysis of the equation governing C. Despite the *ad hoc* nature of the model it has two advantages. First, it illustrates much more directly than the usual approach the essential statistical nature of the dispersion of marked fluid (or indeed of heat transfer). Second, the quantitative agreement between the two approaches is close enough for there to be good prospects of describing real dispersion problems accurately and reasonably efficiently with more sophisticated models.

The velocity of a marked fluid molecule is a random function of time and its longitudinal component has a mean equal to the discharge velocity once the molecule has forgotten where it started from [1]. Throughout this paper axes will be taken moving with the discharge velocity, so that if $X(t)$ is the longitudinal displacement of a marked fluid molecule then $X(t)$ is a random function of time with zero ensemble mean. Further [2], denoting an ensemble mean by an overbar,

$$\frac{d\overline{X^2}}{dt} = 2\int_0^t R(\tau)d\tau\,, \tag{1.1}$$

where $R(\tau)$ is the covariance of the longitudinal velocities of the marked fluid molecule at two times separated by an interval τ, i.e.

$$R(\tau) = \overline{\left\{\frac{dX}{dt}\right\}_{t+\tau}\left\{\frac{dX}{dt}\right\}_t}\,. \tag{1.2}$$

From Eqn. (1.1) it follows [2] that provided $R(\tau)$ tends to zero fast enough as $\tau \to \infty$ then for large t,

$$X^2 \approx 2Dt, \quad D = \int_0^\infty R(\tau)d\tau \,. \tag{1.3}$$

The first of Eqns. (1.3) can also be obtained from an analysis of the equation governing C [3, 4] but now D is given by an integral involving the Eulerian velocity distribution and the lateral eddy diffusivity assuming that the lateral transfer of marked fluid can be described to good approximation by Fick's Law. For two-dimensional flow in a channel bounded by $y=0$ and $y=h$ this integral is [5]:

$$D = \frac{1}{h}\int_0^h \frac{1}{\kappa(y)}\left\{\int_0^y u(y')dy'\right\}^2 dy \,, \tag{1.4}$$

where $u(y)$ and $\kappa(y)$ are the mean longitudinal turbulent velocity and lateral eddy diffusivity respectively.

The statistical model to be investigated in the present paper will be applied to flows taking place in the channel shown in Fig. 1. There are two layers of heights h_1 and h_2 in which the lateral (i.e. in the direction of the y-axis in Fig. 1) eddy diffusivities have the constant values κ_1 and κ_2 respectively. This geometry was chosen because of its simplicity and also, and mainly, because of the availability of results obtained by an analysis of the equation governing C [6], so enabling comparisons to be made. This model was chosen in [6] because it illustrates qualitatively the role of the viscous sub-layer in longitudinal dispersion.

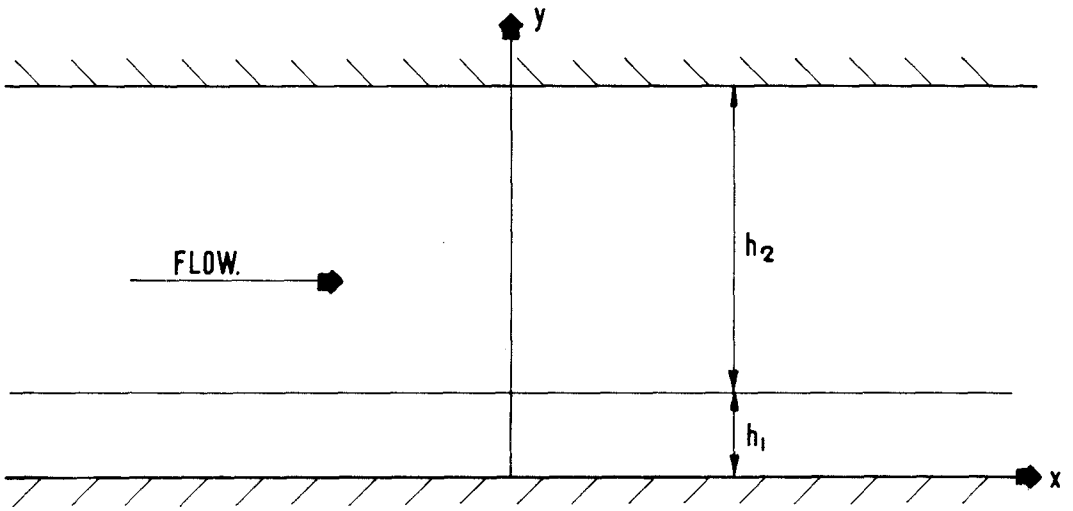

Figure 1. The geometry of the flows analysed in this paper.

2. The structure of the statistical model

In the flow shown in Fig. 1 a marked fluid molecule wanders between $y=0$ and $y=h$. The molecule is said to be in state i when it is in the layer of height h_i. An event of type i occurs whenever the molecule arrives at the interface from state i.

Now the time between any two consecutive events is a random variable with a probability density function $f_i(t)$ where the index i refers to the state the molecule is in between the two events. Notice particularly that $f_i(t)$ is assumed not to depend on the type of the first event. This is equivalent to the assumption that the molecule loses its memory at the interface which seems very reasonable unless the cloud has dimensions of the order of the mean free path of the molecule. The form of $f_i(t)$ is not known but it can depend only on κ_i, h_i and t. For simplicity f_i is here assumed to be exponential so that

$$f_i(t) = (1/t_i)\exp(-t/t_i) \qquad (i=1,2)\,. \tag{2.1}$$

Here t_i is the mean time between the two events. By dimensional arguments

$$t_i = \gamma h_i^2/\kappa_i \qquad (i=1,2)\,, \tag{2.2}$$

where γ is a number expected to be of order one. This choice for f_i implies that the probability of an event of type i occurring between t and $t+\delta t$ is $\delta t/t_i$ independent of when the last event took place provided only that the molecule is in state i at time t.

At an event the molecule either returns to the state from which it came or crosses to the other state. Define α_{ij} by

$$\alpha_{ij} = \Pr\begin{pmatrix}\text{at an event of type } i \text{ the}\\ \text{molecule crosses to state } j\end{pmatrix}, \tag{2.3}$$

Since molecules do not accumulate at the interface

$$\alpha_{11}+\alpha_{12}=1, \quad \alpha_{21}+\alpha_{22}=1\,. \tag{2.4}$$

Now define $p_{ij}(t)$ by

$$p_{ij}(t) = \Pr\left(\begin{matrix}\text{molecule in state}\\ j \text{ at time } t\end{matrix}\;\middle|\;\begin{matrix}\text{event of type}\\ i \text{ at time } 0\end{matrix}\right). \tag{2.5}$$

By a slight generalization of results in [7], section 9.3,

$$\begin{aligned}
p_{11}(t) &= \frac{\alpha_{21}t_1}{\alpha_{21}t_1+\alpha_{12}t_2} + \left(\alpha_{11} - \frac{\alpha_{21}t_1}{\alpha_{21}t_1+\alpha_{12}t_2}\right)e^{-\nu t}\\
p_{21}(t) &= \frac{\alpha_{21}t_1}{\alpha_{21}t_1+\alpha_{12}t_2} + \left(\alpha_{21} - \frac{\alpha_{21}t_1}{\alpha_{21}t_1+\alpha_{12}t_2}\right)e^{-\nu t}\\
p_{12}(t) &= \frac{\alpha_{12}t_2}{\alpha_{21}t_1+\alpha_{12}t_2} + \left(\alpha_{12} - \frac{\alpha_{12}t_2}{\alpha_{21}t_1+\alpha_{12}t_2}\right)e^{-\nu t}\\
p_{22}(t) &= \frac{\alpha_{12}t_2}{\alpha_{21}t_1+\alpha_{12}t_2} + \left(\alpha_{22} - \frac{\alpha_{12}t_2}{\alpha_{21}t_1+\alpha_{12}t_2}\right)e^{-\nu t},
\end{aligned} \tag{2.6}$$

where

$$\nu = \left(\frac{\alpha_{12}}{t_1} + \frac{\alpha_{21}}{t_2}\right). \tag{2.7}$$

Thus as $t\to\infty$,

$$\left.\begin{matrix}p_{11}\\ p_{21}\end{matrix}\right\} \to p_1 = \frac{\alpha_{21}t_1}{\alpha_{21}t_1+\alpha_{12}t_2}\,; \quad \left.\begin{matrix}p_{12}\\ p_{22}\end{matrix}\right\} \to p_2 = \frac{\alpha_{12}t_2}{\alpha_{21}t_1+\alpha_{12}t_2}, \tag{2.8}$$

so that eventually the probability of the molecule being in a particular state is independent of what happened at $t=0$, that is the molecule forgets where it started from.

Thus as $t\to\infty$ all molecules are equivalent so that the probability of a single marked molecule being in a particular state i is by mass conservation proportional to h_i. Hence

$$p_1 = h_1/h, \quad p_2 = h_2/h\,. \tag{2.9}$$

The assumption made above that the molecule loses its memory at the interface implies that α_{ij} does not depend on i. From Eqns. (2.4), (2.8) and (2.9) it follows that

$$\alpha_{12} = 1-\alpha_{11} = \frac{h_1\kappa_2}{h_2\kappa_1+h_1\kappa_2}\,; \quad \alpha_{21} = 1-\alpha_{22} = \frac{h_2\kappa_1}{h_2\kappa_1+h_1\kappa_2}\,. \tag{2.10}$$

From Eqns. (2.6) and (2.10) the following simplified expressions are obtained:

$$p_{11}=p_{21} = \frac{h_1}{h} + \frac{(h_2^2\kappa_1-h_1^2\kappa_2)}{h(h_2\kappa_1+h_1\kappa_2)}e^{-\nu t} \qquad p_{22}=p_{12} = \frac{h_2}{h} + \frac{(h_1^2\kappa_2-h_2^2\kappa_2)}{h(h_2\kappa_1+h_1\kappa_2)}e^{-\nu t} \tag{2.11}$$

It is necessary to introduce one more definition. This is of $P_{ij}(\tau)$ where

$$P_{ij}(\tau) = \Pr\begin{pmatrix}\text{molecule in state } i \text{ at time } t\\ \text{and in state } j \text{ at time } t+\tau\end{pmatrix}, \tag{2.12}$$

and in the notation it has been assumed that t is large enough for the molecule to have forgotten where it started from so that P_{ij} does not depend on t. Under these circumstances

$$\begin{aligned}
P_{ij} &= \Pr\begin{pmatrix}\text{molecule in state}\\ i \text{ at time } t\end{pmatrix} \Pr\left(\begin{array}{l|l}\text{molecule in state} & \text{molecule in state}\\ j \text{ at time } t+\tau & i \text{ at time } t\end{array}\right),\\
&= p_i \Pr\left(\begin{array}{l|l}\text{molecule in state} & \text{molecule in state}\\ j \text{ at time } \tau & i \text{ at time } 0\end{array}\right),\\
&= p_i \left[\Pr\begin{pmatrix}\text{event of type } i \text{ occurs at } \tau' \text{ where } 0<\tau'<\tau\\ \text{and that the molecule is in state } j \text{ at time } \tau\end{pmatrix}\right.\\
&\qquad \left. + \Pr\begin{pmatrix}\text{no event occurs between times}\\ 0 \text{ and } \tau \text{ and that } i=j\end{pmatrix}\right],\\
&= p_i\left[\int_0^\tau p_{ij}(\tau-\tau')f_i(\tau')d\tau' + \delta_{ij}\int_\tau^\infty f_j(\tau')d\tau'\right],
\end{aligned}$$

where δ_{ij} is 1 if $i=j$ and zero otherwise, and summation convention is not used. Using Eqns. (2.1) and (2.11) gives

$$\begin{aligned}
P_{11}(\tau) &= \left(\frac{h_1}{h}\right)^2 + \left(\frac{h_1h_2}{h^2}\right)e^{-\nu\tau}\\
P_{12}(\tau) = P_{21}(\tau) &= \left(\frac{h_1h_2}{h^2}\right)[1-e^{-\nu\tau}]\\
P_{22}(\tau) &= \left(\frac{h_2}{h}\right)^2 + \left(\frac{h_1h_2}{h^2}\right)e^{-\nu\tau}.
\end{aligned} \tag{2.13}$$

Suppose now that when the molecule is in state i it has a longitudinal velocity U_i which is a stationary random function of time whose statistical properties are known and are independent of the lateral motion of the molecule. The mean longitudinal velocity of the molecule is zero provided

$$p_1\overline{U_1} + p_2\overline{U_2} = 0, \tag{2.14}$$

and it will be supposed that axes have been chosen so that this is satisfied. Now Eqns. (1.1) and (1.2) show that $R(\tau)$, the covariance of the longitudinal velocities of the marked molecule at two times separated by an interval τ, is a quantity of importance for describing the dispersion of the cloud. Using the independence of the statistical processes involved, the model introduced here gives

$$\begin{aligned}
R(\tau) = {} & P_{11}(\tau)\overline{U_1(t+\tau)U_1(t)} + P_{12}(\tau)\overline{U_1(t)U_2(t+\tau)}\\
& + P_{21}(\tau)\overline{U_1(t+\tau)U_2(t)} + P_{22}(\tau)\overline{U_2(t+\tau)U_2(t)}.
\end{aligned} \tag{2.15}$$

In simplifying Eqn. (2.15) it will be assumed further that the processes U_1 and U_2 are independent so that

$$\overline{U_1(t+\tau)U_2(t)} = \overline{U_1(t)U_2(t+\tau)} = \overline{U_1}\,\overline{U_2}. \tag{2.16}$$

Thus, using Eqns. (2.13) and (2.14),

$$R(\tau) = P_{11}(\tau)R_1(\tau) + P_{22}(\tau)R_2(\tau) + \left(\frac{h_1h_2}{h^2}\right)(\overline{U_1}-\overline{U_2})^2 e^{-\nu\tau}, \tag{2.17}$$

where $R_1(\tau)$ and $R_2(\tau)$ are the correlation functions for the processes U_1 and U_2 respectively.

This result will be applied to two particular flows and it will be noted that the number of

assumptions already made is such that the only arbitrariness in the model is the value of the constant γ defined in Eqn. (2.2).

3. Two applications of the model

The simplest case is the unrealistic one in which $U_i = \overline{U_i}$ so that $R_1(\tau) = R_2(\tau) = 0$ and, from Eqn. (2.17),

$$R(\tau) = \left(\frac{h_1 h_2}{h^2}\right)(\overline{U_1} - \overline{U_2})^2 \, e^{-\nu\tau} . \tag{3.1}$$

From Eqn. (3.1) the value of the longitudinal diffusivity D, defined in Eqn. (1.3), can be obtained by integration with the result that

$$D = \left(\frac{h_1 h_2}{h^2}\right)\frac{(\overline{U_1} - \overline{U_2})^2}{\nu} = \gamma\left(\frac{h_1^2 h_2^2}{h^3}\right)\left(\frac{h_1}{\kappa_1} + \frac{h_2}{\kappa_2}\right)(\overline{U_1} - \overline{U_2})^2 , \tag{3.2}$$

using Eqns. (2.2), (2.7) and (2.10). On the other hand the value of D can be obtained from Eqn. (1.4) with $\kappa(y) = \kappa_1$ and $u(y) = \overline{U_1}$ for $0 < y < h_1$, and $\kappa(y) = \kappa_2$ and $u(y) = \overline{U_2}$ for $h_1 < y < h_2$. The value obtained is

$$D = \tfrac{1}{3}\left(\frac{h_1^2 h_2^2}{h^3}\right)\left(\frac{h_1}{\kappa_1} + \frac{h_2}{\kappa_2}\right)(\overline{U_1} - \overline{U_2})^2 . \tag{3.3}$$

This is *identical* with Eqn. (3.2) provided

$$\gamma = \tfrac{1}{3} . \tag{3.4}$$

A more severe test of the theory occurs when the $R_i(\tau)$ in Eqn. (2.17) are non-zero so that U_i is a genuine random function. In [6] the case when $u(y)$ had the form

$$u(y) = U(2y/h - 1) \tag{3.5}$$

was analyzed. Here, in applying the statistical model, it is natural to take $\overline{U_i}$ as the average of $u(y)$ with respect to y over the layer of height h_i so that

$$\overline{U_1} = -U(h_2/h), \quad \overline{U_2} = U(h_1/h), \quad \overline{U_1} - \overline{U_2} = -U . \tag{3.6}$$

Further the values of $R_i(\tau)$ can be obtained directly from Eqn. (4.3) of [6] with the result

$$R_i(\tau) = \frac{32U^2}{\pi^4}\left(\frac{h_i}{h}\right)^2 \sum_{n=1}^{\infty} \frac{\exp\{-(2n-1)^2\pi^2\kappa_i\tau/h_i^2\}}{(2n-1)^4} . \tag{3.7}$$

Substituting (3.6) and (3.7) into Eqn. (2.17) gives

$$\begin{aligned} R(\tau) = {} & \frac{32U^2}{\pi^4}\left\{\left(\frac{h_1}{h}\right)^4 + \left(\frac{h_1}{h}\right)^3\left(\frac{h_2}{h}\right)e^{-\nu\tau}\right\}\sum_{n=1}^{\infty}\frac{\exp\{-(2n-1)^2\pi^2\kappa_1\tau/h_1^2\}}{(2n-1)^4} \\ & + \frac{32U^2}{\pi^4}\left\{\left(\frac{h_2}{h}\right)^4 + \left(\frac{h_2}{h}\right)^3\left(\frac{h_1}{h}\right)e^{-\nu\tau}\right\}\sum_{n=1}^{\infty}\frac{\exp\{-(2n-1)^2\pi^2\kappa_2\tau/h_2^2\}}{(2n-1)^4} \\ & + \frac{h_1 h_2}{h^2}\, U^2 e^{-\nu\tau} . \end{aligned} \tag{3.8}$$

The value of D can now be obtained by integration using (1.3) with the result

$$\begin{aligned} D = {} & \frac{U^2 h_1^3}{60h\kappa_1}\left[2\left(\frac{h_1}{h}\right)^3 + 60\gamma\left(\frac{h_2}{h}\right)^2 + 15\left(\frac{h_1}{h}\right)^2\left(\frac{h_2}{h}\right)F(\sqrt{\nu h_1^2/\pi^2\kappa_1})\right. \\ & \left. + \frac{U^2 h_2^3}{60h\kappa_2}\left[2\left(\frac{h_2}{h}\right)^3 + 60\gamma\left(\frac{h_1}{h}\right)^2 + 15\left(\frac{h_2}{h}\right)^2\left(\frac{h_1}{h}\right)F(\sqrt{\nu h_2^2/\pi^2\kappa_2})\right], \end{aligned} \tag{3.9}$$

where

$$F(z) = \frac{128}{\pi^6} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \frac{1}{(2n-1)^2+z^2}. \tag{3.10}$$

It is shown in the Appendix that

$$F(z) = \left\{ \tanh\left(\frac{\pi z}{2}\right) - \left[\left(\frac{\pi z}{2}\right) - \frac{1}{3}\left(\frac{\pi z}{2}\right)^3\right]\right\} \left(\frac{\pi z}{2}\right)^5. \tag{3.11}$$

Now the value of D can be computed in some special cases.

(i) $h_1 = 0$. In this case there is only one layer present and $vh_1^2/\pi^2\kappa = 0$, $vh_2^2/\pi^2\kappa^2 = \infty$ so that, using the relevant properties of $F(z)$ (see the Appendix), Eqn. (3.9) gives

$$D = \frac{U^2h^2}{30\kappa_2}. \tag{3.12}$$

This agrees exactly with the value obtained by use of Eqn. (1.4) as it should, since with only one layer present the value of $R(\tau)$ reduces to the exact value of $R_2(\tau)$ given by (3.7).

(ii) $\kappa_1 = \kappa_2 = \kappa$ and $h_1 \ll h_2$. As far as the geometry is concerned there is again only one layer present, but the derivation of the value of $R(\tau)$ has assumed that the processes U_1 and U_2 are independent which is clearly wrong since, with one layer, there is only one process not two. Thus it can be anticipated that the values of $R(\tau)$ and hence of D given by the model will be less than their true values. Under these conditions

$$\frac{vh_1^2}{\pi^2\kappa_1} \approx \frac{1}{\gamma\pi^2}\left(\frac{h_1}{h_2}\right) \ll 1, \quad \frac{vh_2^2}{\pi^2\kappa_2} \approx \frac{1}{\gamma\pi^2}\left(\frac{h_2}{h_1}\right) \gg 1, \tag{3.13}$$

and then Eqn. (3.9) gives

$$D \approx \frac{U^2h^2}{30\kappa}\left[1 - 6\left(\frac{h_1}{h}\right) + O\left(\frac{h_1}{h}\right)^2\right]. \tag{3.14}$$

As expected this is smaller than the true value which is $U^2h^2/30\kappa$ (see Eqn. (3.12)) but only by a small quantity proportional to h_1/h.

(iii) $h_1 \ll h_2$ and $h_2^2/\kappa_2 \ll h_1^2/\kappa_1$. This is the case considered in detail in [6] where the layer of height h_1 models the viscous sub-layer in a real channel, and the restrictions above model the facts that the viscous sub-layer occupies only a small fraction of the depth of the channel and, notwithstanding this, that the time taken for a marked fluid molecule to sample all parts of the viscous sub-layer is much longer than that taken for it to sample the remaining parts of the channel. With these restrictions, Eqn. (3.9) gives

$$D \approx \frac{U^2h^2}{30}\left[\frac{(30\gamma(h_1/h)^3 + O(h_1/h)^4)}{\kappa_1} + \frac{(1+O(h_1/h))}{\kappa_2}\right]. \tag{3.15}$$

The value obtained from Eqn. (1.4) with the same restrictions is given in Eqn. (3.8) of [6], and is

$$D \approx \frac{U^2h^2}{30}\left[\frac{10(h_1/h)^3}{\kappa_1} + \frac{1}{\kappa_2}\right]. \tag{3.16}$$

To highest order in (h_1/h) the two expressions are the same provided $\gamma = \frac{1}{3}$. It is remarkable that this was also the value of γ necessary for agreement in the previous flow considered (see Eqn. (3.4)).

For case (iii) the value of $R(\tau)$, with the above restrictions, can be obtained approximately from Eqn. (3.8), with the result

$$R(\tau) \approx \frac{32U^2}{\pi^4}\left[\exp\left(\frac{-\pi^2\kappa_2\tau}{h_2^2}\right) + \frac{\pi^4}{32}\left(\frac{h_1}{h}\right)\exp\left(\frac{-\kappa_1\tau}{\gamma h_1^2}\right)\right]. \tag{3.17}$$

The value of $R(\tau)$ obtained in [6] is (see Eqn. (4.2) of that paper)

$$R(\tau) \approx \frac{32U^2}{\pi^4}\left[\exp\left(-\frac{\pi^2\kappa_2\tau}{h_2^2}\right) + \frac{\pi^2}{4}\left(\frac{h_1}{h}\right)\exp\left(-\frac{\pi^2\kappa_1\tau}{4h_1^2}\right)\right]. \tag{3.18}$$

Thus Eqns. (3.17) and (3.18) are not the same (unless $\pi^2 = 8$ *and* $\pi^2 = 4/\gamma$!) but they are very similar. Further the second term in Eqn. (3.17) comes from the last term in Eqn. (2.17), which represents the contribution to $R(\tau)$ arising because the mean velocities in the two layers are different, and not from the second term which derives from the random motion in the sub-layer alone. This conclusion agrees with that made in [6].

4. Conclusions and suggestions for improving the model

Despite the number of assumptions made about the statistical model the results in section 3 show that it is very successful, at least as far as the value of D is concerned, in its predictions, provided γ is chosen as $\frac{1}{3}$. One of the assumptions made is certainly not correct, i.e. that the processes U_1 and U_2 are statistically independent. Nevertheless the error introduced does not appear significant in the cases considered here and it is probable that use of a more realistic hypothesis than independence would reduce the errors still further, although the algebra would be increased.

A second assumption in the model is arbitrary rather than incorrect, and that is the nature of the probability density function $f_i(t)$ of the time between two events. It would be interesting to see how different choices of these functions affected the predictions of the model.

In analysing other flows and geometries by models of the type considered in this paper it would be natural, since $u(y)$ and $\kappa(y)$ do not normally have the simple forms considered here, to divide the cross-section of the pipe up into many layers in each of which $u(y)$ and $\kappa(y)$ are approximately linear and constant respectively.

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Appendix

The purpose of this appendix is to show that $F(z)$ is given by Eqn. (3.11). Write

$$\Sigma_r(z) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^r}\,\frac{1}{(2n-1)^2+z^2} \qquad (r=0, 1, \ldots). \tag{A.1}$$

Then

$$\Sigma_0(z) + z^2\,\Sigma_2(z) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \tag{A.2}$$

and

$$\Sigma_2(z) + z^2\,\Sigma_4(z) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}, \tag{A.3}$$

using well-known results. Thus, using Eqn. (3.10),

$$F(z) = \frac{128}{\pi^6 z^4}\left\{\frac{\pi^4 z^2}{96} - \frac{\pi^2}{8} + \Sigma_0(z)\right\}. \tag{A.4}$$

But, on using partial fractions,

$$-2iz\,\Sigma_0(z) = \sum_{n=1}^{\infty}\left\{\frac{1}{iz+(2n-1)}+\frac{1}{iz-(2n-1)}\right\}$$

$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\left\{\frac{1}{\left(\frac{\pi iz}{2}-\frac{\pi}{2}\right)+n\pi}+\frac{1}{\left(\frac{\pi iz}{2}-\frac{\pi}{2}\right)-(n-1)\pi}\right\}$$

$$=\frac{\pi}{2}\left[\frac{1}{\theta}+2\theta\sum_{n=1}^{\infty}\frac{1}{\theta^2-n^2\pi^2}\right], \tag{A.5}$$

where

$$\theta=\frac{\pi iz}{2}-\frac{\pi}{2}. \tag{A.6}$$

But (see [8]),

$$\cot\theta=\frac{1}{\theta}+2\theta\sum_{n=1}^{\infty}\frac{1}{\theta^2-n^2\pi^2}. \tag{A.7}$$

Thus, from Eqns. (A.5) and (A.6),

$$\Sigma_0(z)=\frac{i\pi}{4z}\cot\left(\frac{\pi iz}{2}-\frac{\pi}{2}\right)$$

$$=\frac{\pi}{4z}\tanh\frac{\pi z}{2}, \tag{A.8}$$

using well-known properties of circular and hyperbolic functions. Thus, substituting in Eqn. (A.4),

$$F(z)=\frac{128}{\pi^6}\left\{\frac{\pi}{4z^5}\tanh\frac{\pi z}{2}-\frac{\pi^2}{8z^4}+\frac{\pi^4}{96z^2}\right\}$$

$$=\left\{\tanh\left(\frac{\pi z}{2}\right)-\left[\left(\frac{\pi z}{2}\right)-\frac{1}{3}\left(\frac{\pi z}{2}\right)^3\right]\right\}\Big/\left(\frac{\pi z}{2}\right)^5, \tag{A.9}$$

which is Eqn. (3.11) as required. The following properties of $F(z)$ are used in the paper:

$$|z|\ll 1 \Rightarrow F(z)=\frac{2}{15}+O(z^2);$$

$$|z|\gg 1 \Rightarrow F(z)=\left(\frac{4}{3\pi^2}\right)z^{-2}+O(z^{-4}). \tag{A.10}$$

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